„K-Theory, vector bundles and characteristic classes“

- Audience: Students of master (oder fortgeschrittener Bachelor)
- prerequesites: algebraic topology
- Dozent: Thomas Schick
- Thema: **K-theory, vector bundles and characteristic classes**
- Time (suggestion) Tue 14:15-15:55
- Place: if possible, in presence (perhaps mix of presence and videoconf)
- credit points for presentation of a talk (with a handout)
- **Preliminary meeting**: Friday Oct 9, 14:15 via videoconference. Please register in stud.ip to participate.
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K-theory is a generalized cohomology theory based on the "cocycles" which are vector bundles. The fact that we really get a generalized cohomology theory (i.e. that the Eilenberg-Steenrod axioms are satisfied) is based on a fundamental result about K-theory: Bott periodicity, which provides a canonical isomorphism between the K-theory of X and $\Sigma^2 X$ (the two-fold suspension, obtained from $S^2 \times X$ by identifying $* \times X$ to one new basepoint).

The power of K-theory is displayed in the fact that it is used as a crucial tool in Adams' proof that the only division algebra structures on finite dimensional real vector spaces exist in dimension 1 ($\mathbb{R}$), 2 ($\mathbb{C}$), 4 (quaternions), and 8 (octonions).

Fundamental in the understanding of vector bundles and of K-theory is the introduction and analysis of the relevant *classifying spaces* $BU(n)$ and BU, most efficiently described as Grassmann manifolds.

(Stable) characteristic classes are transformations from K-theory to ordinary cohomology and widely used to understand vector bundles and K-theory. In particular, we have the Chern classes and the Chern character. On manifolds, there are explicit local formulas to compute them, based on the curvature of a bundle.

The goal of the seminar is:

- introduction to vector bundles
- introduction to K-theory
- proof that K-theory is a cohomology theory
- in particular: proof of Bott theory
- classifying spaces
- application to division algebra structures
- introduction to characteristic classes

The central part of the course will be based on Atiyah's classic book "K-theory", complemented by several further references (e.g. Milnor and Stasheff's "Characteristic classes"). The depth and speed will be determined by the previous knowledge and interest of the participants, as well as the

Program

Thema	Quelle	Name	Termin
Vector bundles	(H) 1.1		
Classification of vector bundles I	(H) 1, (A)		
Classification of vector bundles II	(H) p. 27–31		
K-theory	(H) 39–41, 51–53		
Bott periodicity I	(H) 2		
Bott periodicity II	(H) p. 42–51 or (AB)		
K-theory computations	(H) 65–72		
Adams operations	(H) 2.20, 2.21, p. 70–72		
Division algebras	(H) p. 59–62,65		
Stiefel-Whitney classes and Chern classes	(H) p. 77–83		
Cohomology of Grassmannians	(H) 84–87		
Euler class	(H) p. 88–93, (MS)		
Pontryagin classes	(H) 94–97		
Obstructions to sections	(H) Sec 3.3		

Description of contents

1. **Vector bundles**. Basics: definitions, examples, constructions (direct sum, tensor product, metrics)
2. **Classification of vector bundles I**. pullback bundle, homotopy invariance: (H) Theorem 1.6 or (A) Lemma 1.4.3. Classification of vector bundles on S^n.
3. **Classification of vector bundles II**. Grassmann manifolds and their universial bundles, via them classification of vector bundles on compact spaces
4. **K-theory**. Definition of K-groups, definition of the long exact sequence (H) 51–53
5. **Bott periodicity I**. Fundamental product theory (H) Theorem 2.2. (without proof) and applications (H) Theorem 2.3, p. 53–58
6. **Bott periodicity II**. Proof of fundamental product theory.
7. **K-theory calculations**. Compute $K(\mathbb{C}P^n)$, Leray-Hirsch theorem, Thom isomorphism
8. **Adams operations**. Discuss and prove also the splitting principle.
9. **Division algebras**. Present the Hopf invariant, the Hopf invariant 1 problem, its different formulations (parallelizability of S^n, existence of division algebra structure. Solution of the problem.
10. **Stiefel-Whitney and Chern classes**. "Axiomatic" introduction and construction. First applications.
11. **Cohomology of Grassmanianns**. Computation of the cohomology rings of Grassmannians. Classification of line bundles via first Chern class or first Stiefel-Whitney class
12. **Euler class**. Introduction of Euler class, its relation to the Euler characteristic.
13. **Pontryagin classes**. Definition via Chern classses. Relation to Stiefel-Whitney classes and Euler class. Description of the cohomology of the real Grassmannian in terms of Pontryagin classes.
14. **Obstruction to nowhere vanishing sections**. General theory of obstruction cohomology classes to (extension of) sections

of a bundle. Spezialization to Euler class as primary obstruction for a nowhere vanishing section. If time permits, corresponding interpretation of Stiefel-Whitney classes.

Literature

(A) Atiyah, Michael: K-theory

(AB) Michael Atiyah and Raoul Bott. On the periodicity theorem for complex vector bundles. Acta Math., 112:229–247, 1964.

(H) Hatcher, Alan Vector bundles and K-theory (unfinished book project). http://pi.math.cornell.edu/ hatcher/VBKT/VBpage.html

(MS) Milnor, John and Stasheff, Jim: Characteristic classes

Possible modules:

M.Nat. 4814 (Seminar on algebraic topology); M.Nat.4813 (Seminar on differential geometry); M.Nat.4824 (Seminar on groups, geometry and dynamical systems); M.Nat.4913 (Advanced seminar on differential geometry); M.Nat.4914 (Advanced seminar on algebraic topology); M.Nat.4924 (Advanced seminar on groups, geometry and dynamical systems); M.Nat.4915 (Advanced seminar on mathematical methods in physics); M.Nat 4815 (Seminar on mathematical methods in physcis); B.Mat.3413 (Seminar im Zyklus Differenzialgeometrie); B.Mat.3414 (Seminar im Zyklus Algebraische Topologie); B.Mat.3425 (Seminar im Zyklus Nichtkommutative Geometrie); B.Mat.3424 (Seminar im Zyklus Gruppen, Geometrie und Dynamische Systeme)