Lectures on coarse index theory

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Abstract

Coarse index theory has been introduced by John Roe. It provides a theory to use tools from C^*-algebras to get information about the geometry of non-compact manifolds via index theory of Dirac type operators. Through the passage to the universal covering one also gets important information about compact manifolds, and indeed, the Baum-Connes assembly map becomes part of this theory. In the lecture series, we discuss the basic constructions of the theory and three particular applications: to partitioned manifolds (as introduced by Roe), to enlargeable manifolds (as defined by Gromov and Lawson [1]) and to "the passage from the positive scalar curvature exact sequence to analysis. Along the way, we discuss several new index theorems.

Many of the basics are due to John Roe and Nigel Higson [10] **(Kommentar: Abstract: perhaps remove references, but move then (completed!) to a later place in the introduction)**, the new results are joint with Hanke, Kotschick, Roe [4], Hanke and Pape [5], and with Paolo Piazza [21].

Outline of the course

We first introduce the basic players: we start with a positive dimensional spin manifold X, assign to it the Roe algebras $C^*(X) \subset D^*(X)$. Using an direct and easy construction in C^*-algebras, one defines the Roe-index of the Dirac operator in $K_*(C^*(X))$. If the manifold has positive scalar curvature, this index vanishes. Instead, one gets a secondary invariant $\rho(g) \in K_*(D^*(X))$ which contains information about the positive scalar curvature metric (and can, e.g., distinguish between different components of the space of all such metrics). We will develop this as a recurring theme: a geometric reason for the vanishing of an index should indeed always give rise to a secondary invariant which gives new information about the space.

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Given an isometric action by a discrete group Γ on X, we get algebras of invariants $C^*(X)^\Gamma \subset D^*(X)^\Gamma$ and a refined index/secondary invariant in the K-theory of these algebras. There is an important map induced by "forgetting the equivariance".

We obtain such an isometric action in particular on covering spaces. This is an interesting way to assign non-compact manifolds with extra structure to compact manifolds, and via this route coarse geometry gives information about compact manifolds. Moreover, it is not hard to see that the Baum-Connes assembly map is just a special case of the general constructions of coarse index theory.

The K-theory of $C^*(X)$ and $D^*(X)$ (and there equivariant cousins) admits tools for computation. In particular it is functorial for rather general maps of spaces, and we will establish a Mayer-Vietoris sequence. We will also establish some vanishing results, e.g. for spaces of the form $X \times [0,\infty)$ (with product metric). Crucial here (and a common theme in the application of C^*-algebra techniques) is the implementation of geometric data —here the subspaces into which a space is decomposed— by suitable C^*-ideals.

We will exploit functoriality and the computation machine to explain and refine the obstruction to positive scalar curvature due to "enlargeability" (introduced by Gromov-Lawson) in terms of the standard index obtained via the Baum-Connes assembly map (the Rosenberg or Mishchenko index), with values in the K-theory of the group C^*-algebra (this is joint with Hanke, Kotschick, Roe).

We then obtain, as a first application, the partitioned manifold index and the partitioned manifold index theorem: Whenever X can be partitioned with a two-sided hypersurface M which is compact (or on which Γ acts cocompactly) then the Dirac operator on X defines a partitioned manifold index in $K_{*-1}(C^*(M)^\Gamma)$. The partitioned manifold index theorem (proved for $\Gamma = \{1\}$ by Roe and Higson, for general Γ and even dimension by Esfahani-Zadeh) states that this partitioned manifold index coincides with the one of M. As an application, we obtain that non-vanishing of the index of the hypersurface M is an obstruction to uniformly positive scalar curvature on X.

Next, we will discuss how one can use information like positive scalar curvature given only on part of the manifold. It turns out that C^*-algebraic methods also here apply very efficiently: by an appropriate choice of intermediate C^*-algebras (here, between $C^*(X)$ and the trivial one) one can exploit the additional information and sometimes get vanishing results. As an application we will present a refined obstructions to positive scalar curvature coming from submanifolds of codimension 2, in the spirit of Gromov-Lawson but applicable in more general situations and explaining the theory using C^*-index techniques. This is joint with Hanke and Pape.

Next, we will present a discussion of bordism invariance of the index of the Dirac operator in the language of coarse geometry. As discussed above: a basic feature is the implementation of the geometric situation (here: the manifold, its boundary, but also the relative theory of the manifold relative to its boundary) as ideals within one big algebra (D^*X). Indeed, this approach allows to get a secondary invariant of the bordism (some kind of relative fundamental class) which explains the vanishing of the index of the Dirac operator of a bounding manifold. Although this is certainly well known to the experts, such an approach to bordism invariance does not seem to be part of the available literature.

Finally, we will turn our attention to new index theorems (obtained jointly with Paolo Piazza). The methods allow to define in a canonical way a $C^*(\Gamma)$-index of the Dirac operator for a spin manifold X with boundary and with fundamental group Γ, provided the boundary has positive scalar curvature (and product structure).

We get a secondary Atiyah-Patodi-Singer index theorem for this coarse index: its "delocalized" part, i.e. the image in the K-theory of $D^*\Gamma$, coincides with the rho-invariant of the boundary.

Similarly, we obtain a secondary partitioned manifold index theorem: if X is a partitioned manifold as above with uniformly positive scalar curvature, then one has (as in the primary case) a partitioned manifold rho-invariant.

The index theorem say that this rho-invariant coincides with the rho-invariant of a partitioning hypersurface. We discuss the (surprisingly difficult) proofs of these theorems.

Finally, we give as an application a direct construction, using index theory, of a transformation from the positive-scalar-curvature exact sequence of Stolz to the long exact sequence of the pair $C^* \subset D^*$. This establishes a "map for psc to analysis", in analogy to Higson-Roe's construction of a map from "surgery to analysis". The construction presented here is even more directly using methods from index theory and C^*-algebras.

1 Lecture 1: basic setup of coarse index theory

In the first lecture, we introduce the main players. We start with a Riemannian manifold X of positive dimension, but many of the definitions here generalize to arbitrary metric spaces.

1.1 The Roe algebras $C^*(X) \subset D^*(X)$

The most important main players are: the sequence of C^*-algebras associated to X, $0 \to C^*(X) \to D^*(X) \to D^*(X)/C^*(X) \to 0$. Here, $C^*(X)$ is the so-called *Roe*-algebra of locally compact operators of finite propagation, whereas $D^*(X)$ is the larger algebra of pseudolocal operators of finite propagation.

1.1 Definition. Given a metric space X (e.g. a Riemannian manifold of positive dimension), a $C_0(X)$-module is a Hilbert space X with an action of $C_0(X)$ such that no non-zero element acts as a compact operator. The prototypical example is $H = L^2(E)$, the square summable sections of a Hermitean vector bundle on the manifold X (with action by pointwise multiplication).

For $\phi \in C_0(X)$ we simply write ϕ also for the operator of multiplication by ϕ.

An operator $T \in \mathcal{B}(H)$ has *propagation* $\le R$ if $\phi \circ T \circ \psi = 0$ whenever $\phi, \psi \in C_0(X)$ whose supports have distance at least R.

An $T \in \mathcal{B}(H)$ is called *locally compact* if $T\phi$ and ϕT are compact operators for each $\phi \in C_0(X)$.

$T \in \mathcal{B}(H)$ is called *pseudolocal* if $\phi T \psi$ is compact whenever $\mathrm{supp}(\phi) \cap \mathrm{supp}(\psi) = \emptyset$.

$$D^*(X) := \overline{\{T \in \mathcal{B}(H) \mid T \text{ pseudolocal and of finite propagation}\}}$$
$$C^*(X) := \{T \in D^*(X) \mid T \text{ locally compact}\}.$$

Here, we take the C^*-closure. Note that $C^*(X)$ is an ideal in $D^*(X)$.

1.2 Remark. Strictly speaking, the definition of $C^*(X)$ and $D^*(X)$ depends on the $C_0(X)$-module H. However, by our ampleness condition (no $\phi \in C_0(X)$ acts as a compact operator) any two such algebras are isomorphic (actually, for $D^*(X)$ one has to be a bit more demanding). The isomorphism is not quite canonical, but the induced isomorphism in K-theory is canonical.

1.3 Example. If X is compact then $C^*(X) \cong K(H)$ is isomorphic to the compact operators on a separable Hilbert space.

1.4 Definition. If a discrete group Γ acts isometrically on X (e.g if X is a Γ-covering of a compact manifold X/Γ) we define $C^*(X)^\Gamma \subset D^*(X)^\Gamma$ by only looking at Γ-equivariant operators.

1.2 Functoriality

1.5 Definition. A map $f\colon X \to Y$ is called *coarse* if for each $R > 0$ there is $S > 0$ such that $d(f(x), f(y)) \le S$ whenever $d(x, y) \le R$ and if the inverse image of every bounded subset is bounded ("metric properness").

Note that we do not require continuity!

1.6 Definition. A coarse homotopy between $f, g\colon X \to Y$ is a coarse map $H\colon X \times [0, \infty) \to Y$ such that $H_0 = f$, for each $x \in X$, $H_t(x)$ is constantly equal to $g(x)$ for $t > T(x)$ for a coarse map $T\colon X \to [0, \infty)$ satisfying certain additional requirements.

1.7 Proposition. *Any coarse map $f\colon X \to Y$ induces a homomorphism $C^*(X) \to C^*(Y)$. The induced map in K-theory is canonical and functorial.*

If f is in addition continuous, it induces a homomorphism $D^*(X) \to D^*(Y)$. Again, the induced map in K-theory is canonical and functorial.

$K_*(C^*(f))$ is coarsely homotopy invariant; $K_*(D^*(f))$ is continuously coarsely homotopy invariant.

Proof. The map is obtained by conjugation with a suitable unitary embedding $U\colon H_X \to H_Y$ which *covers* f, i.e. does move the supports approximately as f does. One needs sufficiently ample H_Y to be able to construct such a U. □

1.3 Reduced group C^*-algebra

1.8 Example. If Γ acts freely and cocompactly on X, then there is a K-theory isomorphism

$$C^*_{red}\Gamma \otimes \mathcal{K} \to C^*(X)^\Gamma.$$

The map is induced by the inclusion of one Γ-orbit (identified with $|\Gamma|$) into X. Because Γ acts cocompactly, the inclusion of the orbit is a coarse equivalence and therefore induces a K-theory isomorphism $C^*\,|\Gamma| \to C^*X$. For $|\Gamma|$, we use the $C_0(\Gamma)$-module is $H = l^2(\Gamma) \otimes l^2(\mathbb{N})$. Equivariant finite propagation operators are given by $\mathbb{C}\Gamma \otimes \mathcal{B}$, local compactness requires to use $\mathbb{C}\Gamma \otimes \mathcal{K}$. The norm-closure is by definition $C^*_{red}\Gamma \otimes \mathcal{K}$.

1.9 Remark. It is typically very difficult to map out of $C^*_{red}\Gamma$ or its K-theory (in contrast to $C^*_{max}\Gamma$, which admits by construction a C^*-homomorphism out for each unitary representation of Γ).

Forgetting equivariance: $C^*(X)^\Gamma \to C^*(X)$ (or more concretely $C^*(|\Gamma|)^\Gamma \to C^*(|\Gamma|)$) provides one example of an interesting homomorphism "out". Its target comes from the world of coarse geometry, where efficient calculation tools are available (which do not work for C^*_{red}).

1.4 Paschke duality

1.10 Proposition. *If X is compact then $K_n(D^*(X)/C^*(X)) \cong K_{n-1}(X)$: the K-theory of the quotient C^*-algebra is isomorphic to the topological K-homology of the space X.*

1.11 Proposition. *If X/Γ is compact then $K_n(D^*(X)^\Gamma/C^*(X)^\Gamma) \cong K_{n-1}(X/\Gamma)$: indeed, by a localization principle, up to adding elements of $C^*(X/\Gamma)$, every element of $D^*(X/Gamma)$ can be given arbitrarily small propagation. Then it lifts to a unique equivariant operator in $D^*(X)^\Gamma$. This gives a canonical isomorphism $D^*(X)^\Gamma/C^*(X)^\Gamma \cong D^*(X/\Gamma)/C^*(X/\Gamma)$. Then one applies Proposition 1.10.*

1.5 Index constructions

One of the appealing aspects of coarse index theory are the very direct constructions of index invariants, in the rather general non-compact situation at hand.

1.12 Proposition. *If D is a geometric Dirac type operator on X, and $f\colon \mathbb{R} \to \mathbb{R}$ is smooth and rapidly decaying, then $f(D) \in C^*(X)$.*

1.13 Proposition. *If $\chi\colon \mathbb{R} \to \mathbb{R}$ is a chopping function, i.e. smooth and $\chi(x)-1$ is rapidly decaying for $x \to +\infty$, whereas $\chi(x)+1$ is rapidly decaying for $x \to -\infty$ then $\chi(D) \in D^*(X)$.*

Proof. This and the previous statement follow from unit propagation speed of the wave kernel and elliptic regularity. □

1.14 Definition. Using an odd chopping function $\chi\colon \mathbb{R} \to \mathbb{R}$ we define (if $\dim(X) \equiv 1 \pmod 2$) a *fundamental class* $[D] = [(1+\chi(D))/2] \in K_0(D^*(X)/C^*(X))$. The *index* $\operatorname{ind}(D)$ is then defined as the image under the boundary map in $K_1(C^*(X))$.

1.15 Definition. If M is even dimensional, D is an odd operator on a $\mathbb{Z}/2$-graded vector bundle and we use a suitable unitary U between odd and even spinors (a unitary covering the identity map of X) to define the *fundamental class* $[D] := [U\chi(D)_+] \in K_1(D^*(X)/C^*(X))$. Again, the image under the boundary map is $\operatorname{ind}(D) \in K_0(C^*(X))$.

1.16 Definition. Assume that M is a compact spin manifold with fundamental group Γ and with universal covering X (with cocompact Γ-action) with Dirac operator D. Define the *equivariant fundamental class*

$$[D] \in K_{m+1}(D^*(X)^\Gamma / C^*(X)^\Gamma)$$

and the *Mishchenko index*

$$\alpha(M) := \operatorname{ind}([D]) \in K_m(C^*_{red}\Gamma) = K_m(C^*(X)^\Gamma),$$

using the calculation of Example 1.8. **(Kommentar: Add remarks/definitions to Mishchenko bundle and Mishchenko index.)**

1.17 Remark. Of course, the standard definition of $\alpha(M)$ is as Mishchenko-Fomenko index of the Dirac operator on M twisted with the Mishchenko bundle, or by applying the descent homomorphism to a fundamental class in equivariant KK-theory. It is a well known (but non-trivial) theorem that the coarse construction gives the same result. **(Kommentar: A detailed proof is missing)**.

2 Lecture 2 — First applications: Enlargeability and partitioned manifolds

2.1 rho-classes as secondary invariants

2.1 Definition. If the operator D is invertible (e.g. if D is the spin Dirac operator and X has uniformly positive scalar curvature, so that the Weitzenböck formula implies invertibility) then one can use a chopping function which is identically ± 1 on the spectrum of D. It follows that the above constructions make sense without passing to the quotient D^*/C^*.

We define the (secondary) coarse rho-invariant $\rho(D) := [(1+\chi(D))/2] \in K_0(D^*(X))$ if $\dim(X)$ is odd, and $\rho(D) := [U\chi(D)_+] \in K_1(D^*(X)/C^*(X))$ if $\dim(X)$ is even. This has not the same stability results as the index: it depends on the metric of positive scalar curvature.

If we have a group Γ acting isometrically, then D and the functions of D used above are Γ-equivariant, and we can also choose U Γ-equivariant. Therefore, we get an index in $K_*(C^*(X)^\Gamma)$, and also $\rho(D) \in K_*(D^*(X)^\Gamma)$.

2.2 Corollary. *By homotopy invariance: if we have two metrics of positive scalar curvature on X which are homotopic through metrics of positive scalar curvature in the same bilipschitz class, the rho-invariants of the associated Dirac operators coincide.*

Consequently, $\rho(D)$ can be used to distinguish different components of the space of metrics of positive scalar curvature.

2.3 Remark. A note on K_1: this group is made to classify automorphisms! In particular, cycles for K_1 must be automorphisms of some object. Occasionally, we don't expect this or do not want it, for example if we think of the index of a Fredholm operator $A\colon H_1 \to H_2$ between two different Hilbert spaces. The key point here is that the two Hilbert spaces are isomorphic such that, by composing with such an isomorphism we really get an invertible element in the Calkin algebra of H_1. Moreover, the K_1-element it represents is independent of the choice of the isomorphism $H_2 \to H_1$ (indeed, the space of these is contractible by Kuiper's theorem if H_2 is infinite dimensional).

In general, a morphism between two different objects will only represent a K_1-class if there is a sufficiently canonical (up to effects in K-theory) isomorphism between these two objects. In our situation, this is given by the unitary U. We have to make sure that U is a multiplier of $D^*(X)$. This is guaranteed if U covers the identity map of X in a sufficiently strict sense.

This is somewhat delicate: one has to use the very ample $C_0(X)$-modules we mentioned above to be sure that one can find such a unitary. However, replacing $L^2(S)$ the $\oplus_{n\in\mathbb{N}} L^2(S)$ and D by the diagonal operator which is D in the first summand and the identity in all others, this is possible without changing anything of the construction. **(Kommentar: Make precise the**

ampleness conditions; show why this is giving rise to well defined K-theory classes here.)

2.4 Remark. The construction of $\rho(D)$ reveals a general principle one should use: find the smallest possible subalgebra (depending on the geometric situation) modulo which the construction is permitted, i.e. modulo which $\chi(D)$ is a symmetry. Without any further assumptions, this is $C^*(X)$. If one assumes positive scalar curvature, one can use the ideal 0. We will meet further instances of this principle later.

2.2 Baum-Connes map in coarse index theory

Assume that X has a free cocompact Γ-action. Then we obtain the long exact K-theory sequence for $C^*(X)^\Gamma \to D^*(X)^\Gamma \to D^*(X)^\Gamma / C^*(X)^\Gamma$, which under the identifications possible by Proposition 1.11 and Example 1.8 becomes

$$\cdots \to K_m(C^*_{red}\Gamma) \to K_m(D^*(X)^\Gamma) \to \\ K_{m-1}(X/\Gamma) \xrightarrow{BC} K_{m-1}(C^*_{red}\Gamma) \to \cdots$$

where the map denoted BC indeed is the usual (Baum-Connes) assembly map in topological K-theory. In particular, we can use $X = E\Gamma$ (a contractible free Γ-space) to obtain the usual version of the assembly map which, for torsion-free groups, is conjecturally an isomorphism.

2.3 C^*-rho invariant contains APS and L^2-rho invariant

If Γ is finite, or more generally if Γ contains an element of finite order, the map BC is definitely *not* an isomorphism.

Instead, Higson and Roe [12] show (with significant analytical effort) that there is a non-trivial homomorphism $K_0(D^*(E\Gamma)^\Gamma) \to \mathbb{R}$. If this homomorphism is applied to $\rho(D)$ for a metric of positive scalar curvature on X/Γ one obtains the classical Atiyah-Patodi-Singer rho-invariant of the Dirac operator for the regular representation (the difference of the eta-invariant on X/Γ and the L^2-eta invariant on X (if Γ is finite and therefore X is compact, this is the eta-invariant on X normalized by dividing by $|\Gamma|$).

Note that these APS-rho invariants are one of the most important sources of information which allows us to distinguish metrics of positive scalar curvature (e.g. on lens spaces).

2.4 Enlargeability

2.5 Definition. A compact manifold M of dimension m is called (universally) enlargeable if for every $R > 0$ the universal covering X admits a map $f_R \colon X \to (S^m_R)$ such that

- S^m_R is the sphere with standard metric but rescaled such that $diam(S^m_R) = R$
- f_R is a smooth Lipschitz 1-map (i.e. each differential has norm ≤ 1)
- f_R is compactly supported, i.e. constant outside a compact subset of X
- $\deg(f_R) \neq 0$. Here, the degree is the sum of the local degrees of f at the points of the inverse image of a regular value: a finite sum because f is compactly supported.

For the more general concept of enlargeability (instead of universal enlargeability) one allows any covering, which may also depend on R.

2.6 Example. Every manifold of non-positive sectional curvature is enlargeable. To construct the maps f_R, choose first one compactly supported map $g\colon X \to S^m$ of degree 1 (the usual maps which collapses the complement of a disc to the extra point of the sphere). Compose this map with the radial contraction with factor R/c to the center of the disc to obtain f_R (c is the Lipschitz constant of g).

2.7 Theorem (Gromov-Lawson)**.** *If M is an enlargeable closed spin manifold then M does not admit a metric of positive scalar curvature.*

2.8 Theorem (Hanke-Kotschick-Roe-Schick)**.** *If M is a universally enlargeable spin manifold, then the Rosenberg index $\alpha(M) \neq 0 \in K_m(C^*_{red}\Gamma)$. This implies that M does not admit a metric of positive scalar curvature, but is stronger (by the failure of the Gromov-Lawson-Rosenberg conjecture, exhibited by an example in* [27]*).*

Proof. (1) The *balloon space* B_m is the union of a half-line $[0, \infty)$ with the spheres S^m_k for $k \in \mathbb{N}_{>0}$; where the south pole of S^m_k is identified with $k \in [0, \infty)$ and B_m is given the path metric induced from the given metrics on the building blocks.

Using the Γ-action and composition with a suitable translation, one can arrange that the support of f_k is contained in some $B_z(R_{2k}) \setminus B_z(R_{2k-1})$ where $R_j \geq R_{j-1} + 1$ for each j ($z \in X$ some basepoint).

One can then combine all the maps f_k to one continuous coarse map $f\colon X \to B_m$.

(2) Using Mayer-Vietoris and a decomposition into the union of discs glued along B_{m-1}, one computes $K_*(C^*B_m) \cong K_{*-m}(C^*(B_0))$. An explicit calculation for this simple space then give $K_0(C^*B_0) = \prod_{k\in\mathbb{N}} \mathbb{Z}/\bigoplus_{k\in\mathbb{N}} \mathbb{Z}$ (and $K_1(c^*B_0) = 0$).

(3) The "forget equivariance" homomorphism composed with the map induced by f therefore maps $K_m(C^*_{red}\Gamma) \to K_m(C^* B_m) = \prod_{k\in\mathbb{N}} \mathbb{Z}/\bigoplus_{k\in\mathbb{N}} \mathbb{Z}$, the image of $\alpha(M)$ is an equivalence class $\operatorname{ind}_{B_n}(M)$ of a sequence of integers.

(4) Finally, one uses an L^2-index theorem to obtain a topological formula for the index sequence $\operatorname{ind}_{B_n}(M)$. Indeed, one obtains $\operatorname{ind}_{B_n}(M) = [(\deg(f_1), \deg(f_2), \dots)]$. By assumption, all degrees are non-zero, therefore also $\alpha(M)$.

□

2.5 Mayer-Vietoris

2.9 Definition. Assume $A \subset X$ is a subset of a metric space. We want to replace the inclusion $C^*(A) \hookrightarrow C^*(X)$ by the inclusion of an ideal $C^*(A \subset X) \subset C^*(X)$, but with a K-theory isomorphism $C^*(A) \to C^*(A \subset X)$.

We set $C^*(A \subset X)$ as the C^*-closure of all operators $T \in C^*(X)$ which are *supported near* A, i.e. such that there is $R > 0$ with $T\phi = 0 = \phi T$ whenever $\phi \in C_0(X)$ and $d(\operatorname{supp}(\phi), A) \geq R$.

We define $D^*(A \subset X)$ in the corresponding way.

2.10 Lemma. *The inclusion $C^*(A) \to C^*(A \subset X)$ is a K-theory isomorphism. The inclusion is given by extending the operator on A by 0.*

If $X = A \cup_{\partial A} A \times [0, \infty)$ (as metric space, in particular as Riemannian manifold) then again we get a K-theory isomorphism $D^*(A) \to D^*(A \subset X)$.

The same applies to the algebras of invariant operators, if everything above is Γ-equivariant.

Proof. It follows from coarse invariance of $C^*(A)$ (and coarse homotopy invariance of $D^*(A)$) that the inclusions $C^*(A) \hookrightarrow C^*(U_R(A))$ are K-theory isomorphisms for all A; $D^*(A) \hookrightarrow D^*(U_R(A))$ are K-theory isomorphisms. Then one observes that $C^*(A \subset X)$ is (by definition) the limit of $C^*(U_R(A))$ and $D^*(A \subset X)$ is the limit of $D^*(U_R(A)))$ and applies continuity of K-theory for limits of C^*-algebras. □

2.11 Definition. A decomposition $X = X_1 \cup X_2$ with $X_0 = X_1 \cap X_2$ of metric spaces is called *coarsely excisive* if for each $R > 0$ there is $S > 0$ such that $U_R(X_1) \cap U_R(X_2) \subset U_S(X_0)$.

2.12 Theorem. *For a coarsely excisive decomposition $X = X_1 \cup X_2$ one obtains long exact Mayer-Vietoris sequences*

$$\cdots \to K_n(C^*(X_0 \subset X)) \to K_n(C^*(X_1 \subset X)) \oplus K_n(C^*(X_2 \subset X)) \to \\ \to K_n(C^*(X)) \to K_{n-1}(C^*(X_0 \subset X)) \to \cdots$$

$$\cdots \to K_n(D^*(X_0 \subset X)) \to K_n(D^*(X_1 \subset X)) \oplus K_n(D^*(X_2 \subset X)) \to K_n(D^*(X)) \to K_{n-1}(D^*(X_0 \subset X)) \to \cdots$$

Using the K-theory isomorphisms of Lemma 2.10 we can (and will) replace $C^*(A \subset X)$ by $C^*(A)$ and $D^*(A \subset X)$ by $D^*(A)$ whenever appropriate.

If everything is Γ-invariant, the corresponding result holds for the Γ-invariant algebras.

Proof. The coarse excisiveness implies that $C^*(X_1 \subset X) \cap C^*(X_2 \subset X) = C^*(X_0 \subset X)$. Moreover, the sum of the two ideals is equal to $C^*(X)$ by the finite propagation condition. One then applies the general K-theory Mayer-Vietoris principle.

The same argument applies to D^*. □

2.13 Remark. Because the K-theory of D^* "sees" the topology, $K_*(D^*(A))$ and $K_*(D^*(A \subset X))$ in general will differ significantly! Therefore, we have not established a general Mayer-Vietoris sequence for $K_*(D^*)$. On the other hand, as $K^*(D^*/C^*)$ is K-homology, for which one also has a Mayer-Vietoris sequence, one should expect a Mayer-Vietoris sequence for $K_*(D^*)$ for rather general decompositions. This does not seem to be established in the literature.

2.14 Proposition. *If X is any metric space, then $K_*(C^*(X \times [0,\infty)))$ and $K_*(D^*(X \times [0,\infty)))$ both vanish. Dito with Γ-action.*

Spaces of the form $X \times [0,\infty)$ are examples of flasque *spaces, a concept introduced by Roe. The statement generalizes to arbitrary flasque spaces.*

Proof. This is based on an Eilenberg swindle: the shift to the right (by 1) is a continuous coarse homotopy equivalence and therefore induces the identity on $K_*(D^*)$ and on $K_*(C^*)$.

On the other hand, its inverse (the shift one step to the left) by the finite propagation condition eventually (i.e. after sufficiently many iterations) maps every class to zero (this is a rough argument, the precise construction is quite a bit more complicated). The identity map is equal to 0 only on the zero group and the statement follows. □

2.15 Corollary. $K_*(C^*(\mathbb{R}^n)) \cong K_{*-n}(C^*(pt)) \cong \begin{cases} \mathbb{Z} & *-n \equiv 0 \pmod 2 \\ 0 & *-n \equiv 1 \pmod 2 \end{cases}.$

Proof. Induction and Mayer-Vietoris for the decomposition of $\mathbb{R}^n$ into two half-spaces. □

2.6 Roe's partitioned manifold index

2.16 Definition. Given a discrete group Γ, there is (up to Γ-equivariant homotopy equivalence) a unique contractible free Γ-space $E\Gamma$. It has the universal property that every free Γ-space maps equivariantly to $E\Gamma$ (with a map which is unique up to Γ-homotopy).

It is characterized by the property of being contractible (non-equivariantly).

The quotient $B\Gamma = E\Gamma/\Gamma$ is the associated classifying space.

2.17 Definition. Let X be an m-dimensional spin manifold partitioned by a compact 2-sided hypersurface $Y \subset X$. The signed distance to Y defines a coarse (proper because Y is compact) map $f\colon X \to \mathbb{R}$.

We now define the *partitioned manifold index* as

$$f_*(\mathrm{ind}(D)) \in K_m(C^*\mathbb{R}) = \begin{cases} \mathbb{Z}; & m \equiv 1 \pmod 2 \\ 0; & m \equiv 0 \pmod 2. \end{cases}$$

More generally, if there is an isometric free Γ-action on X preserving Y and now Y/Γ is compact, then we get in addition a classifying map to $E\Gamma$, a contractible space with free Γ-action (the universal covering of $B\Gamma$). We can assume that $B\Gamma$ is compact (otherwise we use some limiting argument). This combines with the previous map to $\mathbb{R}$ to a coarse continuous equivariant map $f\colon X \to \mathbb{R} \times E\Gamma$. Using Mayer-Vietoris then $K_*(C^*(\mathbb{R} \times E\Gamma)^\Gamma) \cong K_{*-1}(C^*(E\Gamma)^\Gamma) \cong K_{*-1}(C^*_{red}\Gamma)$. Then we obtain a Γ-equivariant partitioned manifold index

$$f_*(\mathrm{ind}(D)) \in K_m(C^*(\mathbb{R} \times E\Gamma)^\Gamma) \cong K_{m-1}(C^*_{red}\Gamma).$$

2.18 Theorem (Partitioned manifold index theorem)**.** *In the above situation,*

$$f_*(\mathrm{ind}(D_X)) = \mathrm{ind}(D_Y) \in K_{m-1}(C^*_{red}\Gamma).$$

This means that a global object, spread a priori over all of X, can be read off from the hypersurface Y.

2.19 Corollary. *As* $\mathrm{ind}(D_X)$ *is homotopy invariant, it does depend only mildly on the Riemannian metric on X: as long as one stays in the same bilipschitz-class (so that the identity map is a continuous coarse equivalence), the index doesn't change.*

Therefore, if $\mathrm{ind}(D_Y) \neq 0$*, then D_X can't be invertible, in particular X does not admit a metric of positive scalar curvature within the given bilipschitz class of metrics.*

Proof of Theorem 2.18. The proof consists of two steps:

(1) one shows a cut-and-paste invariance of the partitioned manifold index (this uses general K-theory of C^*-algebra techniques): it is based on the way how the K-theory of $C^*\mathbb{R}$ is computed (by the Mayer-Vietoris principle, there is indeed an inherent concentration of the index information near hypersurfaces).

For this, take the disjoint union with $Y \times \mathbb{R}$, but with map to $\mathbb{R}$ given by the absolute value (i.e. with a silly partition where we put everything on one side). As this factors through a map to $C^*[0,\infty)$ which has zero K-theory, we don't change the partitioned manifold index.

By cut-and-paste invariance, we can then cut off the "positive part of X" and replace it by $Y \times [0,\infty)$ without changing the partitioned manifold index. In a second step, replace also the "negative part of X by $Y \times (-\infty, 0]$ to arrive at $Y \times \mathbb{R}$ with projection map to $\mathbb{R}$.

(2) In the model product case one has now to carry out an explicit calculation and really compute the partitioned manifold index, using e.g. a separation of variable technique. This involves also an explicit understanding of the index map giving rise to the identification of $K_m(C^*(\mathbb{R} \times E\Gamma)^\Gamma)$ with $K_{m-1}(C^*_{red}\Gamma)$.

□

2.20 Remark. The calculation has been carried for $\Gamma = 1$ by Higson [8] (and previously by Roe); Higson's argument has been adapted to general Γ by Esfahani-Zadeh [28], but only for m odd. The case m even still waits to be worked out explicitly.

2.7 Secondary partitioned manifold index theorem

2.21 Definition. Fundamental in our constructions was the free contractible Γ-space $E\Gamma$. We saw that $C^*(E\Gamma)^\Gamma \cong C^*_{red}\Gamma \otimes \mathcal{K}$. Of equal significance is $D^*\Gamma := D^*(E\Gamma)^\Gamma$ and its K-theory; the latter is canonically associated to Γ **(Kommentar: to be precise, one has to approximate $E\Gamma$ by cocompact subspaces and then pass to a limit, to simplify notation we pretend that $E\Gamma$ is cocompact —although in the most interesting situations, namely if Γ contains non-trivial torsion, this can never be the case.)**

2.22 Definition. Assume that X has free Γ-action as before with Γ-invariant partitioning hypersurface Y such that Y/Γ is compact. Assume in addition that X admits a metric of uniformly positive scalar curvature.

We define the partitioned manifold ρ-invariant as

$$f_*(\rho(D)) \in K_{m+1}(D^*(\mathbb{R} \times E\Gamma)^\Gamma) \cong K_m(D^*\Gamma).$$

2.23 Theorem (Secondary partitioned manifold index theorem [21])**.** *Assume that X is as in Definition 2.22 and that, in addition, the metric is of product form near Y (in particular, the metric on Y also has positive scalar curvature, and there is the usual coarse rho-invariant defined for Y). Then we have equality*

- *of the partitioned manifold ρ-invariant of X (with values in $K_m(D^*\Gamma)$)*
- *the image of $\rho(D_Y)$ under the induced map to $E\Gamma$, i.e. also with values in $K_m(D^*\Gamma)$.*

2.24 Corollary. *Assume that we have two metrics of positive scalar curvature on Y/Γ which can be distinguished by different rho-invariants in $K_m(D^*\Gamma)$.*

Then also the product metrics on $Y \times \mathbb{R}$ can not be homotoped into each other through metrics of positive scalar curvature within the given bilipschitz type.

Proof. Again, the proof of the Theorem consists of two steps. The first step is, as in the absolute partitioned manifold index theorem, is a rather soft reduction to the product case, using the finite propagation property and functional calculus.

The second step is surprisingly thorny and technical: it is again an explicit calculation of these secondary indices. We don't know how to do it differently than by using relatively brute force (and precise application e.g. of the definition of the index map in 6-term K-theory sequences).

For the time being, the proof has only be carried out for m even. □

3 Lecture 3: Mapping positive scalar curvature to analysis

3.1 Delocalized coarse Atiyah-Patodi-Singer index theorem

3.1 Proposition. *Let X be a complete Riemannian spin manifold, $Y \subset X$ such that the scalar curvature is uniformly positive outside Y. Let χ be an odd chopping function of absolute value 1 outside a sufficiently close neighborhood of 0. Then $\chi(D)^2 - 1 = 0 \in D^*(X)/C^*(Y \subset X)$.*

Proof. Combination of unit propagation speed and functional calculus with invertibility coming from the (local) Weitzenböck formula. A complete proof can be found in [19]. □

3.2 Corollary. *We get a vanishing theorem for the coarse index:*

Assume X is a complete connected non-compact connected Riemannian spin manifold and there is a compact subset Y such that the scalar curvature is uniformly positive outside Y. Then $\mathrm{ind}(D_X) = 0 \in K_*(C^*X)$.

In particular, if the index is non-zero, we can't have uniformly positive scalar curvature outside a compact set.

Proof. Choose a ray $[0, \infty) =: R \subset X$ (starting at Y). This is possible because X is non-compact, connected and complete.

Then we have the factorization (as Y is compact)

$$K_*(C^*(Y \subset X)) \to K_*(C^*(R \subset X)) \to K_*(C^*X)$$

We know by Proposition 3.1 that $\mathrm{ind}(D_X)$ lifts to $K_*(C^*(Y \subset X))$, but by the vanishing Proposition 2.14 $K_*(C^*(R \subset X) \cong K_*(C^*[0,\infty)) = 0$. Therefore our index, factoring through the zero group, has to vanish.

□

3.3 Proposition. *State and explain the codimension* 2 *obstruction to positive scalar curvature of* [5]*.*

3.4 Definition. Assume that X is a metrically complete m-dimensional Riemannian spin manifold with boundary Y, with a product structure near the boundary. Let $X_\infty := X \cup_Y (Y \times [0,\infty))$ be the same manifold with infinite cylinder attached, and we think of $X \subset X_\infty$.

Moreover, assume that Y has uniformly positive scalar curvature; therefore the same is true for the cylinder part of X_∞.

Then, for a suitable choice of chopping function χ, $\chi(D)^2 - 1 = 0 \in D^*(X_\infty)/C^*(X \subset X_\infty)$ by Proposition 3.1.

Consequently, the index has a canonically defined lift to $K_m(C^*(X \subset X_\infty)) \cong K_m(C^*X)$. This is the coarse APS-index of D (based on invertibility near the boundary).

With Γ-action, the same extends to the Γ-equivariant algebras.

3.5 Theorem (Delocalized coarse APS-index theorem)**.** *In the situation of Definition 3.4, the image of the APS-index in $K_m(D^*X)$ coincides with the image of $\rho(D_Y)$ under the map from $K_m(D^*Y) \to K_m(D^*X)$.*

Dito for the equivariant versions.

Proof. The proof [21] proceeds in two steps.

First, one uses some soft methods, exploiting C^*-techniques, functional calculus, finite propagation speed to replace X by $Y \times \mathbb{R}$, but with an index problem where we compress (spacially) the operator to $Y \times [0,\infty)$.

The second step then consists of a model calculation for this case. This model calculation is exactly the same as for the secondary partitioned manifold index theorem.

As a consequence, the proof so far is complete only for $m \equiv 0 \pmod 2$.

(Kommentar: More details should be given.) □

3.2 The Stolz exact sequence for the classification of positive scalar curvature

Given a reference space X (we should think of $X = B\Gamma$) we consider the following three abelian groups:

- $\Omega_n^{spin}(X)$ consist of compact n-dimensional spin manifolds with a map to X; two such pairs are equivalent, if there is a compact spin manifold W with a map to X whose boundary decomposes $\partial W = M_1 \amalg (-M_2)$ and the maps to X coincide. The group operation is disjoint union. This is a generalized homology group of X: there are good tools to compute it (Mayer-Vietoris sequences, Atiyah-Hirzebruch spectral sequence,...)

- $Pos_n(X)$ consists of compact n-dimensional spin manifolds with positive scalar curvature metric and with reference map to X; two are equivalent if they are bordant; where the bordism W now is required to have a metric of positive scalar curvature (product near the boundary) which restricts to the given metrics on the boundary, we also need to have the reference map to X on W. If we consider the subset with fixed spin manifold M and fixed reference map (e.g. if $X = M$ and we use the identity), this classifies metrics of positive scalar curvature on M up to *bordism*. In some sense, this is the geometrically interesting group: the structure group of metrics of positive scalar curvature.

- $R_{n+1}(X)$ consists of compact $(n+1)$-dimensional spin manifolds with boundary and with reference map to X, and with a given metric of positive scalar curvature on the boundary. The equivalence relation again is bordism (with all the structure!). Using a surgery construction of Gromov-Lawson and Schoen-Yau, it is shown (by Stolz, Hajac) that $R_{n+1}(X)$, despite its geometric definition, only depends on the fundamental group Γ of X, we therefore usually write $R_{n+1}(\Gamma)$. Unfortunately, there is no purely algebraic definition (and consequently no computations) of $R_{n+1}(\Gamma)$. Still: it should be though of as an algebraic correction term. So far, it is not even calculated for $\Gamma = 1$. However, using rho-invariants we do know instances where the group is highly non-trivial (lower bounds, but no upper bounds).

The group structure is always given by disjoint union.

3.6 Theorem. *There is a long exact sequence*

$$\to R_{n+1}(X) \xrightarrow{\partial} Pos_n(X) \to \Omega_n^{spin}(X) \to R_n(X) \to$$

where the maps are the obvious ones: take the boundary of a cycle for R_{n+1} to obtain a cycle for Pos_n, then forget the metric to obtain a cycle for Ω_n^{spin}, then consider a manifold without boundary as a manifold whose boundary is $\emptyset$ (where each point has positive scalar curvature).

Proof. The proof of exactness is a direct consequence of the definitions and an easy exercise. $\square$

3.3 Mapping positive scalar curvature to analysis

One of the culminating results of [21] is:

3.7 Theorem. *There is a commutative diagram of well defined homomorphisms from the Stolz positive scalar curvature sequence to the coarse index theory sequence*

$$\begin{array}{ccccccc}
R_{n+1}(B\Gamma) & \longrightarrow & Pos_n(B\Gamma) & \longrightarrow & \Omega_n^{spin}(B\Gamma) & \longrightarrow & R_n(B\Gamma) \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\alpha} \\
K_{n+1}(C^*(E\Gamma)^\Gamma) & \longrightarrow & K_{n+1}(D^*(E\Gamma)^\Gamma) & \longrightarrow & K_{n+1}(D^*(E\Gamma)^\Gamma/C^*(E\Gamma)^\Gamma) & \longrightarrow & K_n(C^*(E\Gamma)^\Gamma) \\
\downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} \\
K_{n+1}(C^*_{red}\Gamma) & \longrightarrow & K_{n+1}(D^*\Gamma) & \longrightarrow & K_n(B\Gamma) & \longrightarrow & K_n(C^*_{red}\Gamma)
\end{array}$$

We construct the maps α, β, γ as follows:

(1) for α, take a cycle for $R_{n+1}(B\Gamma)$: a compact manifold with boundary of positive scalar curvature and with map to $B\Gamma$. Then pass to the associated Γ-covering with map to $E\Gamma$, take the coarse APS-index as described in Definition 3.4 and push forward to $K_{n+1}(C^*(E\Gamma)^\Gamma)$.

(2) For the map β, take the coarse rho-invariant of the positive scalar curvature metric on the associated Γ-covering as in Definition 2.1 and push it forward to $K_{n+1}(D^*\Gamma)$.

(3) for the map γ, associate to the compact spin manifold the equivariant fundamental class $[D]$ as in Definition 1.16 and push it forward to $K_{n+1}(D^*(E\Gamma)^\Gamma/C^*(E\Gamma)^\Gamma)$.

The non-trivial task now is to check that the maps are well defined *and that the diagram becomes commutative.*

This relies on the secondary partitioned manifold index theorem (giving bordism invariance of the coarse rho-class), on the coarse delocalized APS-index theorem and on further (already known) bordism invariance results for indices.

3.8 Remark. It is our hope that one can map further to some exact sequence of cyclic homology; where finally numerical invariants should live. Because of the structure of $D^*\Gamma$ this is a rather non-trivial task. Even the treatment of the APS rho-invariant by Higson and Roe in [12] required great effort.

3.9 Remark. An older cousin of the Stolz positive scalar curvature exact sequence is the surgery exact sequence, which has the same flavor: the structure set (for surgery of manifold structures on a given homotopy type) is computed in an exact sequence containing as other terms a homology group (with coefficients in the L-theory spectrum) and an algebraic object:

the L-theory of the group ring. Higson and Roe have carried out the program to map this sequence to the coarse geometry exact sequence in [13–15]. In some sense, this is more difficult as one does not have automatic invertibility —which we gain from positive scalar curvature. Higson and Roe use analytic Poincaré duality complexes to achieve their goal.

An interesting project is the construction of such a collection of maps directly from the signature operators; for the structure set this will require an additional perturbation step to obtain the required invertibility (such a perturbation has been carried out in detail in [20].

4 Lecture 4: bordism invariance of coarse index

We have seen that bordism invariance of the (coarse) index is an important structural result. Put it differently: if Y is the spin-boundary of X, then the index of Y vanishes.

Our goal now is twofold:

(1) We want to give a proof of this result which is intrinsic to coarse C^*-methods

(2) we want to implement our philosophy: if there is a geometric reason why an index vanishes, this should actually be implemented by a secondary invariant: here we expect a secondary invariant depending on the bordism X.

Classically, such vanishing results depend on the existence of a relative fundamental class in the relative homology of the pair (X, Y) which is mapped to the fundamental class of Y under a suitable boundary map. This indicates which structure we have to look for: a suitable C^*-ideal which encodes the relative K-homology (in the same way $D^*(X)/C^*(X)$ encode the absolute K-homology of X).

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